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Estimation and prediction in the mixed multiresponse model

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Abstract

In the paper the method for construction of the mixed multiresponse incomplete model, that is the model with fixed and random parameters which characterize the multitrait experiment in which a different subset of the traits is observed on each of the disjoint subsets of experimental units, is considered. This model is transformed to the form of the mixed linear model. The formulas for estimators of fixed parameters and predictors of random parameters with utilization of the REML estimators of the dispersion components are given.

1. Introduction

1.1. Linear mixed model

Let us consider the model of the observed random variables vector **y** in the form

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\mathbf{u} + \mathbf{e} \tag{1.1.1}$$

where **X** is a known matrix of full column rank, **Z** is a known matrix, $\boldsymbol{\beta}$ is the vector of fixed parameters, $\mathbf{u} \sim N(\mathbf{0},\mathbf{G})$, $\mathbf{e} \sim N(\mathbf{0},\mathbf{R})$, **u** and **e** are uncorrelated, and matrices **G** and **R** are nonsingular. Hence $\mathbf{y} \sim N(\mathbf{X}\boldsymbol{\beta},\mathbf{V})$, where $\mathbf{V} = \mathbf{ZGZ}' + \mathbf{R}$.

Let it be possible to write **V** as

$$\mathbf{V} = \sum_{i=1}^{c} \gamma_i \mathbf{V}_i \ , \tag{1.1.2}$$

Key words: mixed model, multiresponse model, best linear unbiased estimator, best linear unbiased predictor, REML method

where γ_i are some unknown parameters, $\mathbf{V}_i$ are known matrices, c is the number of the parameters γ_i.

There are different possible ways of decomposition of the matrix **V**. An example was given by Henderson (1986). He assumed that the vectors **u** and **e** can be correspondingly partitioned into subvectors $\mathbf{u}_i$ and $\mathbf{e}_i$, $i = 1,2,\ldots,k$, such that

$$\mathrm{var}(\mathbf{u}_i) = \mathbf{G}_{ii} g_{ii}\ , \quad \mathrm{cov}(\mathbf{u}_i, \mathbf{u}_j') = \mathbf{G}_{ij} g_{ij}\ ,$$

$$\mathrm{var}(\mathbf{e}_i) = \mathbf{R}_{ii} r_{ii}\ , \quad \mathrm{cov}(\mathbf{e}_i, \mathbf{e}_j') = \mathbf{R}_{ij} r_{ij}\ ,$$

where $\mathbf{G}_{ij}$ and $\mathbf{R}_{ij}$ are known matrices and g_{ij} and r_{ij} are unknown parameters. If such subdivision exists, one can write

$$\mathbf{Zu} = \sum_{i=1}^{k} \mathbf{Z}_i \mathbf{u}_i$$

for some matrices $\mathbf{Z}_i$, $i = 1, 2, \ldots, k$, and

$$\mathbf{V} = \mathbf{Z}\big(\sum_i \mathbf{G}_i^* \theta_i\big)\mathbf{Z}' + \sum_j \mathbf{R}_j^* \alpha_j\ ,$$

where

$$\mathbf{G}_1^* = \begin{bmatrix} \mathbf{G}_{11} & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} \\ . & . & \ldots & . \\ \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} \end{bmatrix}, \quad \mathbf{G}_2^* = \begin{bmatrix} \mathbf{0} & \mathbf{G}_{12} & \cdots & \mathbf{0} \\ \mathbf{G}_{12} & \mathbf{0} & \ldots & \mathbf{0} \\ . & . & \ldots & . \\ \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} \end{bmatrix}, \ldots,$$

$$\mathbf{G}_k^* = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \ldots & \mathbf{G}_{1k} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ . & . & \ldots & . \\ \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} \end{bmatrix}, \quad \mathbf{G}_{k+1}^* = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \mathbf{G}_{22} & \cdots & \mathbf{0} \\ . & . & \ldots & . \\ \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} \end{bmatrix}, \ldots,$$

$$\theta_1 = g_{11}, \quad \theta_2 = g_{12}, \ldots, \theta_k = g_{1k}, \quad \theta_{k+1} = g_{22}, \ldots$$

$$\mathbf{R}_1^* = \begin{bmatrix} \mathbf{R}_{11} & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} \\ . & . & \ldots & . \\ \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} \end{bmatrix}, \quad \mathbf{R}_2^* = \begin{bmatrix} \mathbf{0} & \mathbf{R}_{12} & \cdots & \mathbf{0} \\ \mathbf{R}_{12} & \mathbf{0} & \ldots & \mathbf{0} \\ . & . & \ldots & . \\ \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} \end{bmatrix}, \ldots,$$

$$\mathbf{R}_k^* = \begin{bmatrix} \mathbf{0} & \mathbf{0} \dots & \mathbf{R}_{1k} \\ \mathbf{0} & \mathbf{0} \dots & \mathbf{0} \\ . & . \dots & . \\ \mathbf{0} & \mathbf{0} \dots & \mathbf{0} \end{bmatrix}, \quad \mathbf{R}_{k+1}^* = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \dots \mathbf{0} \\ \mathbf{0} & \mathbf{R}_{22} & \dots \mathbf{0} \\ . & . & \dots . \\ \mathbf{0} & \mathbf{0} & \dots \mathbf{0} \end{bmatrix}, \dots$$

$$\alpha_1 = r_{11}, \quad \alpha_2 = r_{12}, \dots, \alpha_k = r_{1k}, \quad \alpha_{k+1} = r_{22}, \dots .$$

Then $\mathbf{V} = \sum_i \gamma_i \mathbf{V}_i$ with γ_i equal to θ_l or α_k, $\mathbf{V}_i$ equal to $\mathbf{ZG}_l^*\mathbf{Z}'$ or $\mathbf{R}_k^*$, respectively, for adequate i.

1.2. Estimation and prediction

In the model (1.1.1) we are interested in estimation of the vector $\boldsymbol{\beta}$ and prediction of the vector $\mathbf{u}$. In the literature different ways of solving these problems are known. One of them is presented by Henderson et al. (1959). Henderson chooses the estimator of $\boldsymbol{\beta}$ and predictor of $\mathbf{u}$ so that the joint density function of $\mathbf{y}$ and $\mathbf{u}$

$$f(\mathbf{y},\mathbf{u}) = g(\mathbf{y}|\mathbf{u})h(\mathbf{u}) = \text{Const}\cdot \exp\left[-\frac{1}{2}(\mathbf{y}-\mathbf{X}\boldsymbol{\beta}-\mathbf{Zu})'\mathbf{R}^{-1}(\mathbf{y}-\mathbf{X}\boldsymbol{\beta}-\mathbf{Zu})-\frac{1}{2}\mathbf{u}'\mathbf{G}^{-1}\mathbf{u}\right]$$

is maximized. In this way he obtained equations

$$\begin{bmatrix} \mathbf{X}'\mathbf{R}^{-1}\mathbf{X} & \mathbf{X}'\mathbf{R}^{-1}\mathbf{Z} \\ \mathbf{Z}'\mathbf{R}^{-1}\mathbf{X} & \mathbf{Z}'\mathbf{R}^{-1}\mathbf{Z}+\mathbf{G}^{-1} \end{bmatrix} \begin{bmatrix} \boldsymbol{\beta}^o \\ \hat{\mathbf{u}} \end{bmatrix} = \begin{bmatrix} \mathbf{X}'\mathbf{R}^{-1}\mathbf{y} \\ \mathbf{Z}'\mathbf{R}^{-1}\mathbf{y} \end{bmatrix}. \tag{1.2.1}$$

Mao (1982) obtained the same equations in another way. He assumed that the linear predictor of any given linear function of vectors $\boldsymbol{\beta}$ and $\mathbf{u}$ should minimize the variance of prediction error and simultaneously should be unbiased, i.e. expected values of predictor and predictant should be equal. Solutions of (1.2.1) are the following

$$\hat{\mathbf{u}} = \mathbf{GZ}'\mathbf{V}^{-1}(\mathbf{y}-\mathbf{X}\boldsymbol{\beta}^o), \tag{1.2.2}$$

$$\boldsymbol{\beta}^o = (\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}^{-1}\mathbf{y}. \tag{1.2.3}$$

In the case when matrices $\mathbf{G}$ and $\mathbf{R}$ are known, $\boldsymbol{\beta}^o$ is the BLUE of $\boldsymbol{\beta}$ and $\hat{\mathbf{u}}$ is the BLUP of $\mathbf{u}$. Unfortunately, usually $\mathbf{G}$ and $\mathbf{R}$ are unknown.

1.3. REML estimation

If $\mathbf{G}$ and $\mathbf{R}$ are unknown we have to estimate elements of these matrices. One of the possible methods of estimation is the REML method. REML operates on the likelihood of linear functions of the data vector with expectations zero, the so-called error contrasts, or, equivalently, on the part of the likelihood (of the data vector) which is independent of fixed effects. This results in the loss in degrees of freedom due to fitting of fixed effects (Patterson and Thompson, 1971 or Corbeil and Searle, 1976). For $\mathbf{y} \sim N(\mathbf{X\beta}, \mathbf{V})$, the log likelihood is (see e.g. Harville, 1977)

$$\log\mathcal{L} = -(1/2)\,[\text{const} + \log|\mathbf{V}| + \log|\mathbf{X'V^{-1}X}| + (\mathbf{y} - \mathbf{X}\hat{\boldsymbol\beta})'\mathbf{V}^{-1}(\mathbf{y} - \mathbf{X}\hat{\boldsymbol\beta})]\,. \quad (1.3.1)$$

Using matrix equalities given by Harville (1977) and Searle (1979), (1.3.1) can be rewritten as:

$$-2\log\mathcal{L} = \text{const} + \log|\mathbf{R}| + \log|\mathbf{G}| + \log|\mathbf{C}| + \mathbf{y'Py}\,,$$

where $\mathbf{C}$ is the coefficient matrix in (1.2.1) and

$$\mathbf{P} = \mathbf{V}^{-1} - \mathbf{V}^{-1}\mathbf{X}(\mathbf{X'V}^{-1}\mathbf{X})^{-1}\mathbf{X'V}^{-1}\,.$$

There are many ways of finding the parameters vector which minimize the function $-2\log\mathcal{L}$. Meyer (1989) described some of them for special matrices $\mathbf{R}$ and $\mathbf{G}$. In the general case the minimization is usually performed by iterative solving of the set of equations

$$\mathbf{S}\gamma = \mathbf{q}\,,$$

where $c\times c$ matrix $\mathbf{S}$ and c dimensional vector $\mathbf{q}$ have the elements

$$s_{ij} = \text{tr}(\mathbf{PV}_i\mathbf{PV}_j)\,,\quad i,j = 1,2,\ldots,c\,,$$

$$q_i = \mathbf{y'PV}_i\mathbf{Py}\,,\quad i = 1,2,\ldots,c\,,$$

with $\mathbf{V} = \sum_{i=1}^{c}\gamma_i\mathbf{V}_i$. Here tr(**A**) denotes the trace of the matrix **A**.

The estimators of the parameters γ_i obtained by the REML method are quadratic functions of the observations vector, therefore they are even functions of the observation vector. Moreover, they are invariant with respect to expected value of the vector **y** because they do not depend on the vector $\boldsymbol\beta$. Hence $\mathbf{X}\boldsymbol\beta^o$, where $\boldsymbol\beta^o$ is given by (1.2.3) with elements of the matrix **V** estimated by REML method, is unbiased and consistent estimator of $\mathbf{X}\boldsymbol\beta$ and have the asymptotic normal distribution (see Kackar and Harville, 1981 or Mardia and Marshall, 1984).

2. Multiresponse model

The purpose of this chapter is to show how the theory of mixed linear model can be applied for estimation and prediction in the mixed multiresponse model. One of the possible ways is the transformation of the mixed multiresponse model to the form of mixed linear model in which the interpretation of the parameters is preserved.

We will use the following notation:

$\mathbf{A}\otimes\mathbf{B}$ - denotes the Kronecker product of the matrices **A** and **B**,

cs(**A**) (column string of the matrix **A**) is the vector formed from the columns of the matrix **A**,

$$\operatorname*{col}_{i=1}^{t}(\mathbf{A}_i) = [\mathbf{A}'_1, \mathbf{A}'_2, \dots, \mathbf{A}'_t]',$$

$$\operatorname*{diag}_{i=1}^{t}(\mathbf{A}_i) = \begin{bmatrix} \mathbf{A}_1 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_2 & \dots & \mathbf{0} \\ \cdot & \cdot & \dots & \cdot \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{A}_t \end{bmatrix}.$$

The mixed multiresponse model is introduced below similarly as the fixed multiresponse model in the paper by Walkowiak (1987).

Let t be the number of traits observed in the experiment. Let n experimental units be grouped in k disjoint sets such that on all n_i units in the i-th set, $i = 1,\dots,k$, the same t_i traits ($t_i \le t$) are observed. The random variables describing the traits measured on units belonging to the i-th set can be written in the form of $n_i \times t_i$ dimensional matrix $\mathbf{Y}_i$, $i=1,\dots,k$.

The linear model for $\mathbf{Y}_i$ is defined as

$$\mathbf{Y}_i = \mathbf{X}_i\mathbf{B}_i + \mathbf{Z}_i\mathbf{U}\mathbf{M}_i + \mathbf{E}_i \ .$$

In this formula $\mathbf{X}_i$ is a full column rank known matrix, $\mathbf{Z}_i$ is a known matrix, $\mathbf{B}_i$ is the fixed parameters unknown matrix, **U** is $b \times t$ matrix of unobservable random variables common for each set of experimental units, the matrix $\mathbf{M}_i$ is obtained from the identity matrix $\mathbf{I}_t$ by deleting the columns which numbers are the numbers of traits not observed on the i-th set of units, $\mathbf{E}_i$ is the $n_i \times t_i$ matrix of random errors.

Let us assume that each row of the matrix **U** has the distribution $N(\mathbf{0},\boldsymbol{\Sigma}_u)$ and each row of the matrix $\mathbf{E}_i$, $i = 1,2,\dots,k$, has the distribution $N(\mathbf{0}, \mathbf{M}'_i\boldsymbol{\Sigma}_e\mathbf{M}_i)$ where $\boldsymbol{\Sigma}_u$ and $\boldsymbol{\Sigma}_e$ are unknown $t \times t$ positive definite matrices. Moreover, let us assume that rows of the matrices **U** and $\mathbf{E}_i$, $i = 1,2,\dots,k$, are mutually independent within each matrix and between matrices. Above assumptions and equation

$$\operatorname{cs}(\mathbf{ABC}) = (\mathbf{C}'\otimes\mathbf{A}) \operatorname{cs}(\mathbf{B})$$

(Neudecker, 1969) implies the following forms of the expectation and dispersion matrices of the matrix $\mathbf{Y}_i$:

$$\mathrm{E}(\mathbf{Y}_i) = \mathbf{X}_i\mathbf{B}_i\ ,$$

$$\mathrm{D}(\mathbf{Y}_i) = \mathrm{D}(\mathrm{cs}\ (\mathbf{Y}_i)) = (\mathbf{M}_i'\otimes\mathbf{Z}_i)\mathbf{G}(\mathbf{M}_i\otimes\mathbf{Z}_i') + (\mathbf{M}_i'\mathbf{\Sigma}_e\mathbf{M}_i\otimes\mathbf{I}_{n_i})\ ,$$

where

$$\mathbf{G} = \mathrm{D}(\mathrm{cs}\ (\mathbf{U})) = (\mathbf{\Sigma}_u\otimes\mathbf{I}_b) \tag{2.1}$$

is the dispersion matrix of the matrix $\mathbf{U}$.

The general model describing random variables observed in the whole experiment may be constructed by utilizing the method of transforming the multivariate model to a univariate model (Searle, 1978). This model has the form

$$\mathbf{y} = \operatorname*{col}_{i=1}^{k}(\mathrm{cs}(\mathbf{Y}_i)) = \operatorname*{diag}_{i=1}^{k}(\mathbf{I}_{t_i}\otimes\mathbf{X}_i)\operatorname*{col}_{i=1}^{k}(\mathrm{cs}(\mathbf{B}_i)) + \operatorname*{col}_{i=1}^{k}(\mathbf{M}_i'\otimes\mathbf{Z}_i)\mathrm{cs}(\mathbf{U}) + \operatorname*{col}_{i=1}^{k}(\mathrm{cs}(\mathbf{E}_i))$$

or

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\mathbf{u} + \mathbf{e}\ , \tag{2.2}$$

where

$$\mathbf{X} = \operatorname*{diag}_{i=1}^{k}(\mathbf{I}_{t_i}\otimes\mathbf{X}_i)\ ,\quad \boldsymbol{\beta} = \operatorname*{col}_{i=1}^{k}(\mathrm{cs}(\mathbf{B}_i))\ ,\quad \mathbf{Z} = \operatorname*{col}_{i=1}^{k}(\mathbf{M}_i'\otimes\mathbf{Z}_i)\ ,$$

$$\mathbf{u} = \mathrm{cs}(\mathbf{U}),\quad \mathbf{e} = \operatorname*{col}_{i=1}^{k}(\mathrm{cs}(\mathbf{E}_i))\ .$$

The vector of expected values and the dispersion matrix of the vector $\mathbf{y}$ have the following forms:

$$\mathrm{E}(\mathbf{y}) = \mathbf{X}\boldsymbol{\beta}\ ,$$

$$\mathrm{D}(\mathbf{y}) = \mathbf{V} = \mathbf{Z}\mathrm{D}(\mathbf{u})\mathbf{Z}' + \mathrm{D}(\mathbf{e})\ .$$

Then

$$\mathbf{V} = \mathbf{Z}\mathbf{G}\mathbf{Z}' + \mathbf{R}\ , \tag{2.3}$$

where

$$\mathbf{R} = \operatorname*{diag}_{i=1}^{k}(\mathbf{M}_i'\mathbf{\Sigma}_e\mathbf{M}_i\otimes\mathbf{I}_{n_i})\ . \tag{2.4}$$

Our assumptions imply that vectors $\mathbf{y}$, $\mathbf{u}$ and $\mathbf{e}$ have normal distributions. From (2.2) we can see that the above transformations reduce the mixed multiresponse model to the form of mixed model (1.1.1) with special forms of the matrices $\mathbf{X}$, $\mathbf{Z}$ and vectors $\boldsymbol{\beta}$, $\mathbf{u}$ and $\mathbf{e}$.

Now we will demonstrate, that the dispersion matrix (2.3) can be transformed to the form (1.1.2). It is easy to show that there exist real numbers λ_{rq}, $0 \le r < q \le t$, such that

$$\boldsymbol{\Sigma}_u = \sum_{r=0}^{t-1} \sum_{q=r+1}^{t} \lambda_{rq} \boldsymbol{\Sigma}_{rq}, \quad \text{where} \quad \boldsymbol{\Sigma}_{rq} = (\mathbf{e}_r + \mathbf{e}_q)(\mathbf{e}_r + \mathbf{e}_q)', \quad \mathbf{e}_o = \mathbf{0},$$

and $\mathbf{e}_l$ is the t dimensional vector with the l-th coordinate equal to 1 and any other equal to 0. Hence matrix $\mathbf{G}$ given in (2.1) has the form

$$\mathbf{G} = \Big(\sum_{r=0}^{t-1} \sum_{q=r+1}^{t} \lambda_{rq} \boldsymbol{\Sigma}_{rq} \Big) \otimes \mathbf{I}_b = \sum_{r=0}^{t-1} \sum_{q=r+1}^{t} (\boldsymbol{\Sigma}_{rq} \otimes \mathbf{I}_b). \tag{2.5}$$

Similarly, there exist real numbers δ_{rq}, $0 \le r < q \le t$, such that

$$\boldsymbol{\Sigma}_e = \sum_{r=0}^{t-1} \sum_{q=r+1}^{t} \delta_{rq} \boldsymbol{\Sigma}_{rq}$$

and matrix $\mathbf{R}$ defined in (2.4) is the following

$$\mathbf{R} = \operatorname*{diag}_{i=1}^{k}\Big(\mathbf{M}_i'\Big(\sum_{r=0}^{t-1} \sum_{q=r+1}^{t} \delta_{rq} \boldsymbol{\Sigma}_{rq}\Big) \mathbf{M}_i \otimes \mathbf{I}_{n_i}\Big) = \sum_{r=0}^{t-1} \sum_{q=r+1}^{t} \delta_{rq} \operatorname*{diag}_{i=1}^{k}((\mathbf{M}_i'\boldsymbol{\Sigma}_{rq}\mathbf{M}_i) \otimes \mathbf{I}_{n_i}).$$

From (2.5) we have

$$\mathbf{Z}\mathbf{G}\mathbf{Z}' = \sum_{r=0}^{t-1} \sum_{q=r+1}^{t} \lambda_{rq} \mathbf{Z}(\boldsymbol{\Sigma}_{rq} \otimes \mathbf{I}_b)\mathbf{Z}',$$

so that the matrix $\mathbf{V}$ given in (2.3) can be written as

$$\mathbf{V} = \sum_{r=0}^{t-1} \sum_{q=r+1}^{t} \lambda_{rq} \mathbf{Z}(\boldsymbol{\Sigma}_{rq} \otimes \mathbf{I}_b)\mathbf{Z}' + \sum_{r=0}^{t-1} \sum_{q=r+1}^{t} \delta_{rq} \operatorname*{diag}_{i=1}^{k}((\mathbf{M}_i'\boldsymbol{\Sigma}_{rq}\mathbf{M}_i) \otimes \mathbf{I}_{n_i}) =$$

$$= \sum_{r=0}^{t-1} \sum_{q=r+1}^{t} \lambda_{rq} [\mathbf{Z}((\mathbf{e}_r + \mathbf{e}_q) \otimes \mathbf{I}_b)][\mathbf{Z}((\mathbf{e}_r + \mathbf{e}_q) \otimes \mathbf{I}_b)]' +$$

$$+ \sum_{r=0}^{t-1} \sum_{q=r+1}^{t} \delta_{rq} [\operatorname*{diag}_{i=1}^{k}(\mathbf{M}_i'(\mathbf{e}_r + \mathbf{e}_q) \otimes \mathbf{I}_{n_i})][\operatorname*{diag}_{i=1}^{k}(\mathbf{M}_i'(\mathbf{e}_r + \mathbf{e}_q) \otimes \mathbf{I}_{n_i})]'.$$

Hence

$$\mathbf{V} = \sum_{j=1}^{c} \gamma_j \mathbf{V}_j \ , \tag{2.6}$$

where

$$c = 2t(t+1)\,/\,2 = t(t+1)\ ,$$

$$\gamma_1 = \lambda_{01},\ \gamma_2 = \lambda_{02}, \ldots, \gamma_{t(t+1)/2} = \lambda_{(t-1)t}\ ,$$

$$\gamma_{(t(t+1)/2)+1} = \delta_{01}, \ldots, \gamma_{t(t+1)} = \delta_{(t-1)t}\ ,$$

$$\mathbf{V}_1 = [\mathbf{Z}((\mathbf{e}_o + \mathbf{e}_1) \otimes \mathbf{I}_b)][\mathbf{Z}((\mathbf{e}_o + \mathbf{e}_1) \otimes \mathbf{I}_b)]' \ ,$$

....

$$\mathbf{V}_{t(t+1)/2} = [\mathbf{Z}((\mathbf{e}_{t-1} + \mathbf{e}_t) \otimes \mathbf{I}_b)][\mathbf{Z}((\mathbf{e}_{t-1} + \mathbf{e}_t) \otimes \mathbf{I}_b)]' \ ,$$

$$\mathbf{V}_{(t(t+1)/2)+1} = [\operatorname*{diag}_{i=1}^{k}(\mathbf{M}_i'(\mathbf{e}_o + \mathbf{e}_1) \otimes \mathbf{I}_{n_i})][\operatorname*{diag}_{i=1}^{k}(\mathbf{M}_i'(\mathbf{e}_o + \mathbf{e}_1) \otimes \mathbf{I}_{n_i})]' \ ,$$

....

$$\mathbf{V}_c = [\operatorname*{diag}_{i=1}^{k}(\mathbf{M}_i'(\mathbf{e}_{t-1} + \mathbf{e}_t) \otimes \mathbf{I}_{n_i})][\operatorname*{diag}_{i=1}^{k}(\mathbf{M}_i'(\mathbf{e}_{t-1} + \mathbf{e}_t) \otimes \mathbf{I}_{n_i})]' \ .$$

From the last formulas we can see that matrices $\mathbf{V}_j$, $j = 1,2,\ldots,c$, are nonnegative definite. Therefore, and because of (2.6), the considered multiresponse model belongs to the known in the literature (see e.g. Kala, 1981) class of general linear models.

According to the theory given in earlier paragraphs unbiased linear estimator of $\boldsymbol{\beta}$ is given as in (1.2.3), i.e.

$$\boldsymbol{\beta}^o = (\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}^{-1}\mathbf{y}$$

and unbiased linear predictor of $\mathbf{u}$ is given as in (1.2.2), i.e.

$$\hat{\mathbf{u}} = \mathbf{G}\mathbf{Z}'\mathbf{V}^{-1}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}^o)\ .$$

In order to estimate unknown matrices $\mathbf{G}$ and $\mathbf{R}$, parameters γ_j in (2.6) can be estimated utilizing the REML method described in section 1.3.

The special case of the theory described in this paper is illustrated by numerical example given by Moliński et al. (1993).

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Estymacja i predykcja w mieszanym modelu wieloreakcjowym

Streszczenie

W pracy opisana jest metoda konstrukcji mieszanego modelu wieloreakcjowego niekompletnego, tzn. modelu z parametrami stałymi i losowymi, opisującego wielocechowy eksperyment, w którym różne podzbiory cech obserwowane są na rozłącznych podzbiorach jednostek eksperymentalnych. Wyprowadzone są także wzory estymatorów parametrów stałych i predyktorów parametrów losowych z zastosowaniem estymatorów komponentów wariancji obliczonych metodą REML.

Słowa kluczowe: model mieszany, model wieloreakcjowy, najlepszy liniowy nieobciążony estymator (BLUE), najlepszy liniowy nieobciążony predyktor (BLUP), metoda REML.